PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 24, Fasc. 1 (2004), pp. 1–15

ON ANNUITIES UNDER RANDOM RATES OF INTEREST WITH PAYMENTS VARYING IN ARITHMETIC AND GEOMETRIC PROGRESSION

BY

K. BURNECKI*, A. MARCINIUK AND A. WERON* (WROCŁAW)

Abstract. In the article we consider accumulated values of annuities-certain with yearly payments with independent random interest rates. We focus on general annuities with payments varying in arithmetic and geometric progression which are important varying annuities. We derive, via recursive relationships, mean and variance formulae of the final values of the annuities. Special cases of our results correct main outcome of Zaks [4].

Mathematics Subject Classification: 91B28, 91B70.

Key words and phrases: Annuity, accumulated value, random interest rate.

1. INTRODUCTION

An *annuity* is defined as a sequence of payments of a limited duration which we denote by n (see, e.g., Gerber [2]). The accumulated or final values of annuities are of our interest. Typically, for simplicity, it is assumed that underlying interest rate is fixed and the same for all years. However, the interest rate that will apply in future years is of course neither known nor constant. Thus, it seems reasonable to let interest rates vary in a random way over time, cf. e.g. Kellison [3].

We assume that annual rates of interest are independent random variables with common mean and variance. We apply this assumption in order to compute, via recursive relationships, fundamental characteristics, namely mean and variance, of the accumulated values of annuities with payments varying in arithmetic and geometric progression. These important varying annuities can be reduced to the cases considered by Zaks [4].

In Section 2 we introduce basic principles of the theory of annuities. Under the fixed interest assumption we consider accumulated values of stan-

* Research supported in part by KBN Grant No. PBZ KBN 016/P03/1999.

dard and non-standard annuities. Finally, we introduce the ones with payments varying according to arithmetic and geometric progression. It appears that all important types of annuities (cf. Kellison [3]) can be obtained as examples of the introduced ones.

In Section 3 we drop the assumption of fixed interest rates and we study the final values of the varying payment annuities under stochastic approach to interest. We consider annual rates of interest to be independent random variables with common mean and variance. Using recursive relations we compute the first and second moment as well as variance of the accumulated values. Special cases of the derived results correct main outcome of Zaks [4], which was pointed out in Burnecki et al. [1].

In Section 4 we illustrate variance results comparing them with computer approximations obtained by means of the Monte Carlo method.

2. ANNUITIES UNDER FIXED INTEREST RATES

First let us recall basic notation used in the theory of annuities. Suppose that j is a positive annual interest rate and fixed through the period of n years. The *annual discount rate* d is given by the formula

$$(1+j)d = j \tag{1}$$

and the *annual discount factor* v is given by the equation

$$(1+j)v = 1. \tag{2}$$

Hence we have

$$v+d = 1. \tag{3}$$

In the article we concentrate on final or accumulated values of annuities. We assume that $k \leqslant n$ throughout, unless otherwise specified. The *accumulated value of an annuity-due* with k annual payments of 1 is denoted by $\ddot{s}_{\overline{k}|j}$ and given by the formulae

$$\ddot{s}_{\overline{k}|j} = \frac{(1+j)^k - 1}{d} \tag{4}$$

and

$$\ddot{s}_{\overline{k}|j} = (1+j)^k + (1+j)^{k-1} + \ldots + (1+j) = (1+j)(1+\ddot{s}_{\overline{k-1}|j}), \tag{5}$$

where the latter defines the recursive equation for $\ddot{s}_{\overline{k}|j}$.

Let us now consider a standard increasing annuity-due. The accumulated value of such an annuity with k annual payments of 1, 2, …, k, respectively, is

$$(I\ddot{s})_{\overline{k}|j} = \frac{\ddot{s}_{\overline{k}|j} - k}{d}. \tag{6}$$

The *accumulated value of an increasing annuity-due* with k annual payments of $1^2, 2^2, \ldots, k^2$, respectively, is denoted by $(I^2\ddot{s})_{\overline{k}|j}$ and calculated by the formula

(7) $$(I^2\ddot{s})_{\overline{k}|j} = \frac{2(I\ddot{s})_{\overline{k}|j} - \ddot{s}_{\overline{k}|j} - k^2}{d}.$$

The following two equations give the recursive formula for $(I^2\ddot{s})_{\overline{k}|j}$ and set the relationship between $(I^2\ddot{s})_{\overline{k}|j}$ and $\ddot{s}_{\overline{k}|j}$:

(8) $$(I^2\ddot{s})_{\overline{k}|j} = (1+j)^k + 2^2(1+j)^{k-1} + \ldots + k^2(1+j) = (1+j)\big(k^2 + (I^2\ddot{s})_{\overline{k-1}|j}\big),$$

(9) $$(I^2\ddot{s})_{\overline{k}|j} = \frac{(1+v)(\ddot{s}_{\overline{k}|j} + k^2) - 2k - 2k^2}{d^2}.$$

The latter corrects Corollary 2.1 from Zaks [4].

In the sequel we will need the following two relations:

(10) $$(I\ddot{s})_{\overline{k-1}|j} = (I\ddot{s})_{\overline{k}|j} - \ddot{s}_{\overline{k}|j},$$

(11) $$(I^2\ddot{s})_{\overline{k-1}|j} = (I^2\ddot{s})_{\overline{k}|j} - 2(I\ddot{s})_{\overline{k}|j} + \ddot{s}_{\overline{k}|j}.$$

Standard decreasing annuities are similar to increasing ones, but the payments are made in the reverse order. The accumulated value of such an annuity-due with k annual payments of $n, n-1, \ldots, n-k+1$, respectively, is denoted by $(D\ddot{s})_{\overline{n,k}|j}$ and given by the formulae:

(12) $$(D\ddot{s})_{\overline{n,k}|j} = n(1+j)^k + (n-1)(1+j)^{k-1} + \ldots + (n-k+1)(1+j)$$
$$= (1+j)\big((D\ddot{s})_{\overline{n,k-1}|j} + (n-k+1)\big)$$

and

(13) $$(D\ddot{s})_{\overline{n,k}|j} = (n+1)\ddot{s}_{\overline{k}|j} - (I\ddot{s})_{\overline{k}|j}.$$

The sum of a standard increasing annuity and its corresponding standard decreasing annuity is of course a constant annuity.

Now let us consider the accumulated value of an annuity-due with payments varying in arithmetic progression (see, e.g., Kellison [3]). The first payment is p and they subsequently increase by q per period, i.e. they form a sequence $(p, p+q, p+2q, \ldots, p+(k-1)q)$. We note that p must be positive but q can be either positive or negative as long as $p+(k-1)q > 0$ in order to avoid negative payments. The accumulated value of such an annuity will be denoted by $(\ddot{s}_a)^{(p,q)}_{\overline{k}|j}$ and is defined by

(14) $$(\ddot{s}_a)^{(p,q)}_{\overline{k}|j} = p(1+j)^k + (p+q)(1+j)^{k-1} + \ldots + \big(p+(k-1)q\big)(1+j).$$

Simple calculations lead to the following relationship:

(15) $$(\ddot{s}_a)^{(p,q)}_{\overline{k}|j} = (p-q)\ddot{s}_{\overline{k}|j} + q(I\ddot{s})_{\overline{k}|j}.$$

Important special cases are the combinations of $p = 1$ and $q = 0$, $p = 1$ and $q = 1$, and $p = n$ and $q = -1$.

EXAMPLE 2.1. *If $p = 1$ and $q = 0$, then $(\ddot{s}_a)^{(p,q)}_{\overline{k}|j}$ becomes the accumulated value of an annuity-due with k annual payments of* 1, *namely*

$$(\ddot{s}_a)^{(1,0)}_{\overline{k}|j} = \ddot{s}_{\overline{k}|j}. \tag{16}$$

EXAMPLE 2.2. *If $p = 1$ and $q = 1$, then $(\ddot{s}_a)^{(p,q)}_{\overline{k}|j}$ becomes the accumulated value of an increasing annuity-due with k annual payments of* 1, 2, ..., *k, respectively, namely*

$$(\ddot{s}_a)^{(1,1)}_{\overline{k}|j} = (I\ddot{s})_{\overline{k}|j}. \tag{17}$$

EXAMPLE 2.3. *If $p = n$ and $q = -1$, then $(\ddot{s}_a)^{(p,q)}_{\overline{k}|j}$ becomes the accumulated value of a decreasing annuity-due with k annual payments of $n, n-1, \ldots, n-k+1$, respectively, namely*

$$(\ddot{s}_a)^{(n,-1)}_{\overline{k}|j} = (D\ddot{s})_{\overline{n,k}|j}. \tag{18}$$

Let us finally consider the accumulated value of an annuity-due with k annual payments varying in geometric progression (see, e.g., Kellison [3]). The first payment is p and they subsequently increase in geometric progression with common ratio q $(q \neq 1+j)$ per period, i.e. they form a sequence $(p, pq, pq^2, pq^3, \ldots, pq^{k-1})$. We note that p and q must be positive in order to avoid negative payments. The accumulated value of such an annuity will be denoted by $(\ddot{s}_g)^{(p,q)}_{\overline{k}|j}$ and is expressed as

$$(\ddot{s}_g)^{(p,q)}_{\overline{k}|j} = p(1+j)^k + pq(1+j)^{k-1} + pq^2(1+j)^{k-2} + \ldots + pq^{k-1}(1+j) = p(1+j)\frac{(1+j)^k - q^k}{1+j-q}. \tag{19}$$

Important special cases are the combinations of $p = 1$ and $q = 1$ (cf. Example 2.1), and $p = 1$ and $q = 1+u$, where u $(u \neq j)$ denotes a fixed rate of increase of the payments.

EXAMPLE 2.4. *If $p = 1$ and $q = 1$, then $(\ddot{s}_g)^{(p,q)}_{\overline{k}|j}$ becomes the accumulated value of an annuity-due with k annual payments of* 1, *namely*

$$(\ddot{s}_g)^{(1,1)}_{\overline{k}|j} = \ddot{s}_{\overline{k}|j}. \tag{20}$$

EXAMPLE 2.5. *If $p = 1$ and $q = 1+u$, then $(\ddot{s}_g)^{(p,q)}_{\overline{k}|j}$ becomes the accumulated value of an annuity-due with k annual payments of* $1, 1+u, (1+u)^2, \ldots, (1+u)^{k-1}$, *respectively. Moreover, it is easy to see that*

$$(\ddot{s}_g)^{(1,1+u)}_{\overline{k}|j} = \frac{(1+j)^k}{1+t}\ddot{s}_{\overline{k}|t}, \tag{21}$$

where t is defined as the solution of

$$1+u = (1+j)(1+t). \tag{22}$$

3. ANNUITIES UNDER RANDOM INTEREST RATES

Let us suppose that the annual rate of interest in the kth year is a random variable i_k. We assume that, for each k, we have $E(i_k) = j > 0$ and $\mathrm{Var}(i_k) = s^2$, and that $i_1, i_2, \ldots, i_n$ are independent random variables. We write

$$E(1+i_k) = 1+j = \mu \tag{23}$$

and

$$E[(1+i_k)^2] = (1+j)^2+s^2 = 1+f = m, \tag{24}$$

where

$$f = 2j+j^2+s^2. \tag{25}$$

Obviously,

$$\mathrm{Var}(1+i_k) = m-\mu^2. \tag{26}$$

Next we define r to be the solution of

$$1+r = \frac{1+f}{1+j}, \tag{27}$$

and using (25) we have

$$r = j+\frac{s^2}{1+j}. \tag{28}$$

For a k-year variable annuity-due with annual payments of $c_1, c_2, \ldots, c_k$, respectively, we denote their final value by C_k.

3.1. Payments varying in arithmetic progression. In the case of payments varying in arithmetic progression we have $c_k = p+(k-1)q$, where $k = 1, 2, \ldots, n$.

The final value of an annuity with such payments is given recursively:

$$C_k = (1+i_k)[C_{k-1}+(p+(k-1)q)] \quad \text{for } k = 2, \ldots, n. \tag{29}$$

We can easily find $\mu_k = E(C_k)$ as

$$\begin{aligned} E(C_k) &= E\big((1+i_k)[C_{k-1}+(p+(k-1)q)]\big) \\ &= E(1+i_k)\,E\big(C_{k-1}+(p+(k-1)q)\big) \end{aligned} \tag{30}$$

from independence of interest rates. Thus we have the recursive equation for $k = 2, \ldots, n$:

$$\mu_k = \mu[\mu_{k-1}+(p+(k-1)q)]. \tag{31}$$

We note that $\mu_1 = p(1+j) = p\mu$. The following lemma stems from (31) and (14).

Lemma 3.1. *If C_k denotes the final value of an annuity-due with annual payments varying in arithmetic progression: $p, p+q, p+2q, \ldots, p+(k-1)q$, respectively, and if the annual rate of interest during the kth year is a random variable i_k such that $E(1+i_k) = 1+j$ and $\operatorname{Var}(1+i_k) = s^2$, and $i_1, i_2, \ldots, i_n$ are independent random variables, then*

$$\mu_k = E(C_k) = (\ddot{s}_a)^{(p,q)}_{\overline{k}|j}. \tag{32}$$

Similarly, for the second moment $E(C_k^2)$ we have the recursive equation for $k = 2, \ldots, n$:

$$m_k = E(C_k^2) = m\left[m_{k-1} + 2(p+(k-1)q)\mu_{k-1} + (p+(k-1)q)^2\right]. \tag{33}$$

We note that $m_1 = p^2 m$. In order to compute the second moment we need the following lemma.

Lemma 3.2. *Under the assumptions of Lemma* 3.1 *we have*

$$m_k = M_{1k} + 2M_{2k}, \tag{34}$$

where

$$M_{1k} = p^2 m^k + (p+q)^2 m^{k-1} + \ldots + (p+(k-1)q)^2 m \tag{35}$$

and

$$M_{2k} = (p+q)m^{k-1}\mu_1 + (p+2q)m^{k-2}\mu_2 + \ldots + (p+(k-1)q)m\mu_{k-1}. \tag{36}$$

Proof. We proceed by induction. When $k = 2$, this follows on the equation (33), since $\mu_1 = p(1+j) = p\mu$ and $m_1 = p^2 m$. Assuming our result is true for a given k ($2 \leqslant k \leqslant n-1$), it stems from formula (33) that it is also true for $k+1$. This concludes the proof by induction. ∎

Since, by (23), $1+f = m$, we can easily find that

$$M_{1k} = p^2 \ddot{s}_{\overline{k}|f} + 2pq(I\ddot{s})_{\overline{k-1}|f} + q^2(I^2\ddot{s})_{\overline{k-1}|f}. \tag{37}$$

Now we can apply (10) and (11) in order to derive an equivalent expression for M_{1k}.

Lemma 3.3. *We have*

$$M_{1k} = (p-q)^2 \ddot{s}_{\overline{k}|f} + 2q(p-q)(I\ddot{s})_{\overline{k}|f} + q^2(I^2\ddot{s})_{\overline{k}|f}. \tag{38}$$

Now we shall determine M_{2k} using (15), (36) and the fact that $1+f = m$. Writing

$$\begin{aligned} M_{2k} = {} & (p+q)(1+f)^{k-1}\left[(p-q)\ddot{s}_{\overline{1}|j} + q(I\ddot{s})_{\overline{1}|j}\right] \\ & + (p+2q)(1+f)^{k-2}\left[(p-q)\ddot{s}_{\overline{2}|j} + q(I\ddot{s})_{\overline{2}|j}\right] + \ldots \\ & + (p+(k-1)q)(1+f)\left[(p-q)\ddot{s}_{\overline{k-1}|j} + q(I\ddot{s})_{\overline{k-1}|j}\right] \end{aligned} \tag{39}$$

$$= \frac{d(p-q)+q}{d^2}\Big[\big((p+q)(1+f)^{k-1}(1+j)$$

$$+(p+2q)(1+f)^{k-2}(1+j)^2+\ldots+(p+(k-1)q)(1+f)(1+j)^{k-1}\big)$$

$$-\big((p+q)(1+f)^{k-1}+(p+2q)(1+f)^{k-2}+\ldots$$

$$+(p+(k-1)q)(1+f)\big)\Big]-\frac{q}{d}\Big[(p+q)(1+f)^{k-1}$$

$$+2(p+2q)(1+f)^{k-2}+\ldots+(k-1)\big(p+(k-1)q\big)(1+f)\Big]$$

and applying (27) we obtain the following results.

LEMMA 3.4. *Under the assumptions of Lemma* 3.1 *we have*

$$M_{2k} = \frac{1}{d^2}\Big[(p-q)\big(d(p-q)+q\big)(1+j)^k \ddot{s}_{\overline{k}|r} + q\big(d(p-q)+q\big)(1+j)^k (I\ddot{s})_{\overline{k}|r}$$
$$-(p-q)\big(d(p-q)+qv\big)\ddot{s}_{\overline{k}|f} - q\big(2d(p-q)+qv\big)(I\ddot{s})_{\overline{k}|f} - q^2 d(I^2\ddot{s})_{\overline{k}|f}\Big]. \tag{40}$$

LEMMA 3.5. *Under the assumptions of Lemma* 3.1 *we have*

$$m_k = \frac{1}{d^2}\Big[(q-p)\big(d(p-q)(1+v)+2qv\big)\ddot{s}_{\overline{k}|f} - 2q\big(d(p-q)(1+v)+qv\big)(I\ddot{s})_{\overline{k}|f}$$
$$-dq^2(1+v)(I^2\ddot{s})_{\overline{k}|f} + 2(p-q)\big(d(p-q)+q\big)(1+j)^k \ddot{s}_{\overline{k}|r}$$
$$+2q\big(d(p-q)+q\big)(1+j)^k (I\ddot{s})_{\overline{k}|r}\Big]. \tag{41}$$

We have thus reached a formula for $E(C_k^2)$. In order to compute $\mathrm{Var}(C_k)$ we need yet an expression for $E(C_k)^2$.

LEMMA 3.6. *Under the assumptions of Lemma* 3.1 *we have*

$$\mu_k^2 = \frac{p-q}{d}\left(p-q+\frac{2q}{d}\right)(\ddot{s}_{\overline{2k}|j} - 2\ddot{s}_{\overline{k}|j}) - \frac{2q(p-q)k}{d}\ddot{s}_{\overline{k}|j}$$
$$+\left(\frac{q}{d}\right)^2\big((I\ddot{s})_{\overline{2k}|j} - 2(1+kd)(I\ddot{s})_{\overline{k}|j} - k^2\big). \tag{42}$$

Proof. It is easy to show that

$$(\ddot{s}_{\overline{k}|j})^2 = \frac{\ddot{s}_{\overline{2k}|j} - 2\ddot{s}_{\overline{k}|j}}{d} \tag{43}$$

and

$$(I\ddot{s})^2_{\overline{k}|j} = \frac{(I\ddot{s})_{\overline{2k}|j} - 2(1+kd)(I\ddot{s})_{\overline{k}|j} - k^2}{d^2}, \tag{44}$$

cf. Lemmas 3.3 and 4.3 from Zaks [4]. From (15) we may write

$$\begin{aligned}\mu_k^2 &= \left((p-q)\,\ddot{s}_{\overline{k}|j}+q\,(I\ddot{s})_{\overline{k}|j}\right)^2 \\ &= (p-q)^2\,(\ddot{s}_{\overline{k}|j})^2+2q\,(q-p)\,\ddot{s}_{\overline{k}|j}\,(I\ddot{s})_{\overline{k}|j}+q^2\,(I\ddot{s})^2_{\overline{k}|j}.\end{aligned} \tag{45}$$

Substituting from (43), (44) and (6) completes the proof. ■

Now, we are allowed to state the following theorem.

THEOREM 3.1. *Under the assumptions of Lemma* 3.1 *we have*

$$E(C_k) = (\ddot{s}_a)^{(p,q)}_{\overline{k}|j}, \tag{46}$$

$$\operatorname{Var}(C_k) = m_k-\mu_k^2, \tag{47}$$

where m_k is given by Lemma 3.5, *and μ_k^2 by Lemma* 3.6.

Let us now consider the situation when $p = 1$ and $q = 0$. We know, from Example 2.1, that it is the case of an annuity-due with k annual payments of 1. Then we obtain the following corollary, cf. Zaks [4].

COROLLARY 3.1. *If C_k denotes the final value of an annuity-due with k annual payments of* 1 *and if the annual rate of interest during the kth year is a random variable i_k such that $E(1+i_k) = 1+j$ and $\operatorname{Var}(1+i_k) = s^2$, and $i_1, i_2, \ldots, i_n$ are independent random variables, then*

$$E(C_k) = \ddot{s}_{\overline{k}|j}, \tag{48}$$

$$\operatorname{Var}(C_k) = \frac{2\,(1+j)^{k+1}\,\ddot{s}_{\overline{k}|r}-(2+j)\,\ddot{s}_{\overline{k}|f}-(1+j)\,\ddot{s}_{\overline{2k}|j}+2\,(1+j)\,\ddot{s}_{\overline{k}|j}}{j}. \tag{49}$$

Another important case is the combination of $p = 1$ and $q = 1$, see Example 2.2. It is an annuity-due with k annual payments of $1, 2, \ldots, k$. The following corollary is a direct consequence of Lemmas 3.1 and 3.3–3.5, cf. Burnecki et al. [1].

COROLLARY 3.2. *If C_k denotes the final value of an increasing annuity-due with k annual payments of $1, 2, \ldots, k$, respectively, and if the annual rate of interest during the kth year is a random variable i_k such that $E(1+i_k) = 1+j$ and $\operatorname{Var}(1+i_k) = s^2$, and $i_1, i_2, \ldots, i_n$ are independent random variables, then*

(a) $E(C_k) = (I\ddot{s})_{\overline{k}|j}$,

(b) $M_{1k} = (I^2\ddot{s})_{\overline{k}|f}$,

(c) $$M_{2k} = \frac{(1+j)^{k+2}\,(I\ddot{s})_{\overline{k}|r}-(1+j)\,(I\ddot{s})_{\overline{k}|f}-j\,(1+j)\,(I^2\ddot{s})_{\overline{k}|f}}{j^2},$$

(d) $$m_k = \frac{2\,(1+j)^{k+2}\,(I\ddot{s})_{\overline{k}|r}-2\,(1+j)\,(I\ddot{s})_{\overline{k}|f}-j\,(2+j)\,(I^2\ddot{s})_{\overline{k}|f}}{j^2}.$$

These results can be summarized in the following corollary, cf. Burnecki et al. [1].

COROLLARY 3.3. *Under the assumptions of Corollary* 3.2 *we have*

$$E(C_k) = (I\ddot{s})_{\overline{k}|j}, \tag{50}$$

$$\operatorname{Var}(C_k) = \frac{2(1+j)^{k+2}(I\ddot{s})_{\overline{k}|r} - 2(1+j)(I\ddot{s})_{\overline{k}|f} - j(2+j)(I^2\ddot{s})_{\overline{k}|f}}{j^2} - \frac{(I\ddot{s})_{\overline{2k}|j} - 2(1+kd)(I\ddot{s})_{\overline{k}|j} - k^2}{d^2}. \tag{51}$$

Let us finally consider the situation when $p = n$ and $q = -1$, see Example 2.3. Then we obtain the following corollary, cf. Burnecki et al. [1].

COROLLARY 3.4. *If* C_k *denotes the final value of a decreasing annuity-due with k annual payments of* $n, n-1, \ldots, n-k+1$, *respectively, and if the annual rate of interest during the kth year is a random variable* i_k *such that* $E(1+i_k) = 1+j$ *and* $\operatorname{Var}(1+i_k) = s^2$, *and* $i_1, i_2, \ldots, i_n$ *are independent random variables, then*

$$E(C_k) = (D\ddot{s})_{\overline{n,k}|j}, \tag{52}$$

$$\operatorname{Var}(C_k) = \frac{l}{d^2}\Bigg[\frac{(n-1/j)^2(1+j)^{2k}\ddot{s}_{\overline{k}|l}}{1+l} - \frac{2(n-1/j)^2(1+j)^k\ddot{s}_{\overline{k}|r}}{1+r} + \frac{(n-1/j)^2\ddot{s}_{\overline{k}|f}}{1+f} + \frac{2(n-1/j)(1+j)^k(I\ddot{s})_{\overline{k}|r}}{1+r} - \frac{2(n-1/j)(I\ddot{s})_{\overline{k}|f}}{1+f} + \frac{(I^2\ddot{s})_{\overline{k}|f}}{1+f}\Bigg], \tag{53}$$

where $l = \big(s/(1+j)\big)^2$.

3.2. Payments varying in geometric progression. In the case of annuities-due with payments varying in geometric progression we have $c_k = pq^{k-1}$, where $k = 1, 2, \ldots, n$. We assume that p and q are positive, $q \neq 1+j$, $q^2 \neq 1+f$ and $q \neq 1+r$.

The final value of that annuity is given recursively:

$$C_k = (1+i_k)[C_{k-1} + pq^{k-1}]. \tag{54}$$

As in the case of payments varying in arithmetic progression, we easily find that for $k = 2, \ldots, n$

$$\mu_k = E(C_k) = \mu[\mu_{k-1} + pq^{k-1}]. \tag{55}$$

The second moment $E(C_k^2)$ is given by

$$m_k = E(C_k^2) = m[m_{k-1} + 2pq^{k-1}\mu_{k-1} + p^2q^{2(k-1)}]. \tag{56}$$

We note that $\mu_1 = p(1+j) = p\mu$ and $m_1 = p^2 m$. By analogy with Lemma 3.1 we obtain a pleasing form of $E(C_k)$.

LEMMA 3.7. *If C_k denotes the final value of an annuity-due with k annual payments varying in geometric progression: $p, pq, pq^2, \ldots, pq^{k-1}$, respectively, and if the annual rate of interest during the kth year is a random variable i_k such that $E(1+i_k) = 1+j$ and $\mathrm{Var}(1+i_k) = s^2$, and $i_1, i_2, \ldots, i_n$ are independent random variables, then*

$$\mu_k = E(C_k) = (\ddot{s}_g)_{\overline{k}|j}^{(p,q)}. \tag{57}$$

As in the previous subsection, in order to find a formula for $\mathrm{Var}(C_k)$ we are about to compute m_k and μ_k^2. We commence by calculating m_k.

LEMMA 3.8. *Under the assumptions of Lemma 3.7 we have*

$$\begin{aligned} m_k = {} & p^2 m^k + p^2 q^2 m^{k-1} + \ldots + p^2 q^{2(k-1)} m \\ & + 2\left[pqm^{k-1}\mu_1 + pq^2 m^{k-2}\mu_2 + \ldots + pq^{k-1} m\mu_{k-1}\right]. \end{aligned} \tag{58}$$

Proof. The assertion follows by induction, applying (56) and the fact that $1+f = m$. ∎

Let

$$M_{1k} = p^2 m^k + p^2 q^2 m^{k-1} + \ldots + p^2 q^{2(k-1)} m \tag{59}$$

and

$$M_{2k} = pqm^{k-1}\mu_1 + pq^2 m^{k-2}\mu_2 + \ldots + pq^{k-1} m\mu_{k-1}. \tag{60}$$

Hence

$$m_k = M_{1k} + 2M_{2k} \tag{61}$$

(cf. Lemma 3.2). Since $1+f = m$, we can easily obtain an elegant expression for M_{1k}.

LEMMA 3.9. *We have*

$$M_{1k} = p^2(1+f)\frac{(1+f)^k - q^{2k}}{1+f-q^2} = (\ddot{s}_g)_{\overline{k}|f}^{(p^2,q^2)}. \tag{62}$$

Now we rewrite (60) applying $1+f = m$ and $1+f = (1+j)(1+r)$, giving

$$\begin{aligned} M_{2k} = {} & pq(1+f)^{k-1} p(1+j)\frac{(1+j)-q}{1+j-q} \\ & + pq^2(1+f)^{k-2} p(1+j)\frac{(1+j)^2 - q^2}{1+j-q} + \ldots \\ & + pq^{k-1}(1+f) p(1+j)\frac{(1+j)^{k-1} - q^{k-1}}{1+j-q} \\ = {} & \frac{p^2(1+j)}{1+j-q}\left[\left(q(1+f)^{k-1}(1+j) + q^2(1+f)^{k-2}(1+j)^2 + \ldots\right.\right. \end{aligned} \tag{63}$$

$$+q^{k-1}(1+f)(1+j)^{k-1}+(1+f)^k-(1+f)^k)$$
$$-\left(q^2(1+f)^{k-1}+q^4(1+f)^{k-2}+\ldots+q^{2(k-1)}(1+f)+(1+f)^k-(1+f)^k\right)]$$
$$=\frac{p^2(1+j)}{1+j-q}\left[(1+j)^k(1+r)\frac{(1+r)^k-q^k}{1+r-q}-\frac{(\ddot{s}_g)_{\overline{k}|f}^{(p^2,q^2)}}{p^2}\right].$$

Therefore we may write the following lemma.

LEMMA 3.10. *We have*

$$M_{2k}=\frac{p(1+j)^{k+1}(\ddot{s}_g)_{\overline{k}|r}^{(p,q)}-(1+j)(\ddot{s}_g)_{\overline{k}|f}^{(p^2,q^2)}}{1+j-q}. \tag{64}$$

By virtue of Lemmas 3.9 and 3.10, and the fact that $m_k = M_{1k}+2M_{2k}$, we have the following lemma.

LEMMA 3.11. *Under the assumptions of Lemma* 3.7 *we have*

$$m_k=\frac{2p(1+j)^{k+1}(\ddot{s}_g)_{\overline{k}|r}^{(p,q)}-(q+1+j)(\ddot{s}_g)_{\overline{k}|f}^{(p^2,q^2)}}{1+j-q}. \tag{65}$$

Thus, we have reached a formula for $E(C_k^2)$. Now we need to derive an expression for $E(C_k)^2$.

LEMMA 3.12. *Under the assumptions of Lemma* 3.7 *we have*

$$\mu_k^2=\frac{p(1+j)}{1+j-q}\left((\ddot{s}_g)_{\overline{2k}|j}^{(p,q)}-2q^k(\ddot{s}_g)_{\overline{k}|j}^{(p,q)}\right). \tag{66}$$

Proof. From Lemma 3.7 and (19) we have

$$\mu_k^2=\left(p(1+j)\frac{(1+j)^k-q^k}{1+j-q}\right)^2 \tag{67}$$
$$=p^2(1+j)^2\frac{(1+j)^{2k}-2(1+j)^kq^k+q^{2k}}{(1+j-q)^2}$$
$$=\frac{p^2(1+j)^2}{1+j-q}\left(\frac{(1+j)^{2k}-q^{2k}}{1+j-q}-\frac{2q^k((1+j)^k-q^k)}{1+j-q}\right),$$

which using (19) completes the proof. ∎

Since $\mathrm{Var}(C_k)=m_k-\mu_k^2$, we have the following theorem.

THEOREM 3.2. *Under the assumptions of Lemma* 3.7 *we have*

$$\mathrm{Var}(C_k)=\frac{2p(1+j)^{k+1}(\ddot{s}_g)_{\overline{k}|r}^{(p,q)}-(1+j+q)(\ddot{s}_g)_{\overline{k}|f}^{(p^2,q^2)}}{1+j-q} -\frac{p(1+j)\left((\ddot{s}_g)_{\overline{2k}|j}^{(p,q)}-2q^k(\ddot{s}_g)_{\overline{k}|j}^{(p,q)}\right)}{1+j-q}. \tag{68}$$

An important case (see Example 2.4) is the combination of $p = 1$ and $q = 1$. Then we obtain an annuity-due with k annual payments of 1 and Theorem 3.2 yields Corollary 3.1.

Another important case is the combination of $p = 1$ and $q = 1+u$, where u $(u \neq j)$ denotes a fixed rate of increase of the payments. This defines an annuity-due with k annual payments of $1, 1+u, (1+u)^2, \ldots, (1+u)^{k-1}$, respectively (see Example 2.5). We assume also that $1+u = (1+j)(1+t)$, $1+f = (1+u)^2(1+h)$ and $1+f = (1+j)^2(1+t)(1+w)$. This leads to the following corollary, cf. Burnecki et al. [1].

COROLLARY 3.5. *If C_k denotes the final value of an annuity-due with k annual payments of* $1, 1+u, (1+u)^2, \ldots, (1+u)^{k-1}$, *respectively, and if the annual rate of interest during the kth year is a random variable i_k such that* $E(1+i_k) = 1+j$ *and* $\operatorname{Var}(1+i_k) = s^2$, *and $i_1, i_2, \ldots, i_n$ are independent random variables, then*

$$E(C_k) = (\ddot{s}_g)_{\overline{k}|j}^{(1,1+u)} = \frac{(1+j)^k \ddot{s}_{\overline{k}|t}}{1+t},$$

$$\operatorname{Var}(C_k) = \frac{(1+u)^{2k}(2+t)\,\ddot{s}_{\overline{k}|h} - 2(1+j)^{2k}(1+t)^k\,\ddot{s}_{\overline{k}|w}}{t} - \frac{(1+j)^{2k}(\ddot{s}_{\overline{2k}|t} - 2\ddot{s}_{\overline{k}|t})}{t(1+t)}.$$

4. FINDING NUMERICAL SOLUTION

In this section we approximate mean and variance of the final values of different annuities applying numerical approach. This tool can be very useful for verifying analytical results. The procedure is as follows (cf. Kellison [3]).

(1) Make an appropriate assumption about the probability function for i_k. This uniquely defines the parameters j and s^2.

(2) Using standard simulation techniques compute m sets of values for $i_1, i_2, \ldots, i_k$.

(3) For each of the m sets $i_1, i_2, \ldots, i_k$ compute the required accumulated value.

(4) The m outcomes are used to compute sample mean and variance.

As a result we obtain an approximation for $E(C_k)$ and $\operatorname{Var}(C_k)$. We may compare them with analytical results.

In order to apply the procedure let us assume that random variables i_k have common normal distribution with parameters $\mu = 0.08$ and $\sigma = 0.02$. This yields that $j = 0.08$ and $s^2 = 0.0004$. Moreover, we set $n = 10$ and $m = 100\,000$. We shall focus on the variance results. We will plot $\mathrm{Var}(C_k)$ as a function of k for three different types of annuities, discussed also by Zaks [4], using the analytical and numerical outcomes.

Figure 1 depicts variance results for an increasing annuity (see Example 2.2). In the left panel we can see the graph of the sample and analytical,

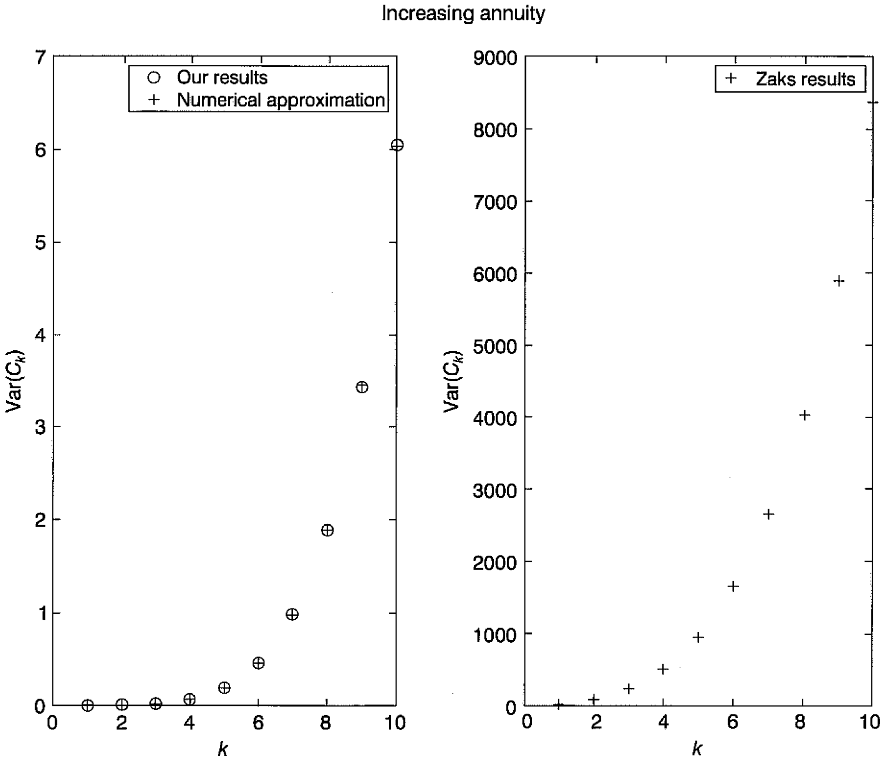

Fig. 1. Comparison of the analytical (+) and numerical (o) results on variance of the final value of an increasing annuity-due. The right panel applies to Theorem 4.3 from Zaks [4] and the left one to Corollary 3.3.

obtained via Corollary 3.3, $\mathrm{Var}(C_k)$. Markedly, the outcomes coincide. The right panel presents the results in the light of Theorem 4.3 from Zaks [4]. Evidently, now the numbers are approximately 1000 times bigger.

Similarly, Figure 2 depicts the comparison for a decreasing annuity (see Example 2.3). The left panel presents the outcomes obtained by means of numerical approximation and of Corollary 3.4. As in the previous case, the results tally. The graph in the right panel of Figure 2 was plotted using the formula from Theorem 4.5 by Zaks [4]. We can clearly see that the results differ dramatically from the ones presented in the left panel.

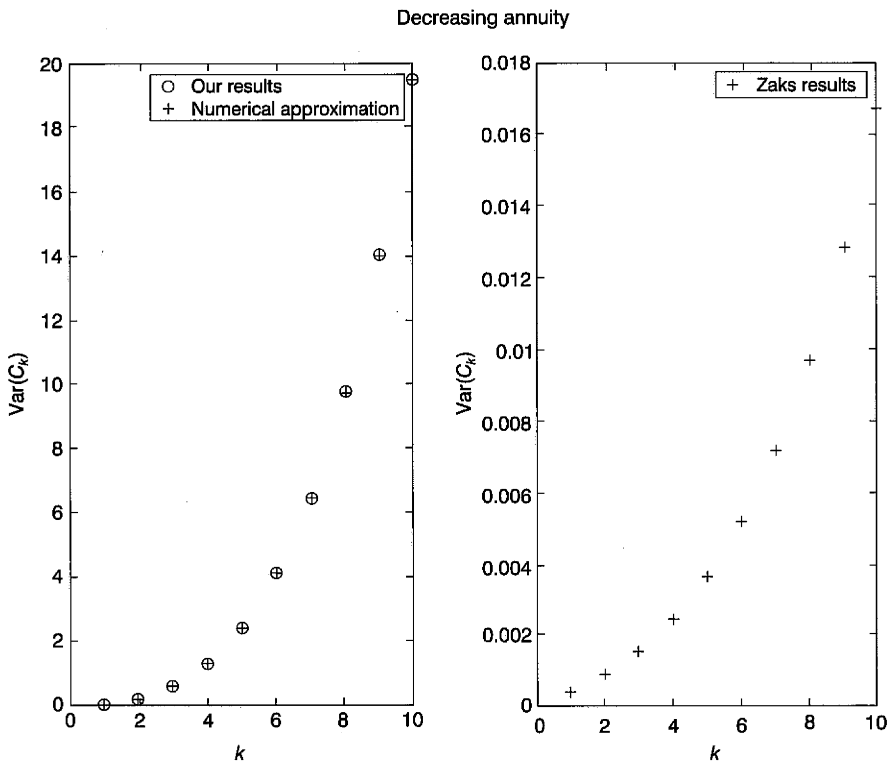

Fig. 2. Comparison of the analytical (+) and numerical (o) results on variance of the final value of a decreasing annuity-due. The right panel applies to Theorem 4.5 from Zaks [4] and the left one to Corollary 3.4.

Finally, Figure 3 depicts the comparison for an annuity with payments varying in geometric progression with $p = 1$ and $q = 1+u$ (see Example 2.5), where we set $u = 0.1$. As before, we can observe in the left panel that our analytical (see Corollary 3.5) and numerical results agree while the corresponding Theorem 4.6 from Zaks [4] yields outcome which is essentially smaller (right panel). It is even negative for $k = 1$.

We conducted similar tests for general annuities with payments varying in arithmetic and geometric progression (see Theorems 3.1 and 3.2). Since they do not have an equivalence in the paper by Zaks [4], we only compared analytical outcomes with numerical approximations. The results have always, as in the foregoing special cases, coincided.

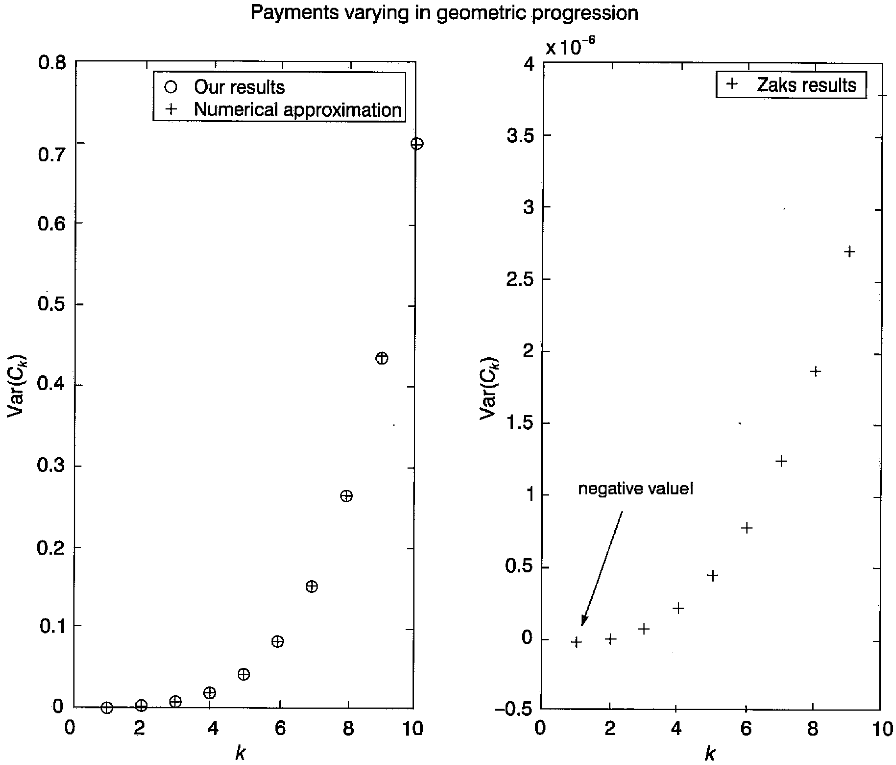

Fig. 3. Comparison of the analytical (+) and numerical (o) results on variance of the final value of an annuity-due with payments varying in geometric progression with $p = 1$ and $q = 1+u$. The right panel applies to Theorem 4.6 from Zaks [4] and the left one to Corollary 3.5. See also Burnecki et al. [1].

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Krzysztof Burnecki and Aleksander Weron
Hugo Steinhaus Center for Stochastic Methods
Institute of Mathematics
Wrocław University of Technology
50-370 Wrocław, Poland
E-mail: burnecki@im.pwr.wroc.pl
weron@im.pwr.wroc.pl

Agnieszka Marciniuk
Department of Statistics
and Economic Cybernetics
Wrocław University of Economics
53-345 Wrocław, Poland
E-mail: aga_marciniuk@tlen.pl

Received on 10.7.2003